Spherical Couette flow of Oldroyd 8-constant model Part I. Solution up to the second-order approximation

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Abstract: The steady flow of an incompressible Oldroyd 8-constant fluid in the annular region between two spheres, or so-called spherical Couette flow, is investigated. The inner sphere rotates with an angular velocity Ω about the z-axis, which passes through the center of the spheres, while the outer sphere is kept at rest. The viscoelasticity of the fluid is assumed to dominate the inertia such that the latter can be neglected in the momentum equation. An analytical solution is obtained through the expansion of the dynamical variables in a power series of the dimensionless retardation time. The leading velocity term denotes the Newtonian rotation about the z-axis. The first-order term results in a secondary flow represented by the stream function that divides the flow region into four symmetric parts. The second-order term is the viscoelastic contribution to the primary viscous flow. The first-order approximation depends on the viscosity and four of the material time-constants of the fluid. The second-order approximation depends on the eight viscometric parameters of the fluid. The torque acting on the outer sphere has an additional term due to viscoelasticity that depends on all the material parameters of the fluid under consideration. For an Oldroyd-B fluid this contributed term enhances the primary torque but in the case of fluids with higher elasticity the torque components may be enhanced or diminished depending on the values of the viscometric parameters.

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Résumé : Nous étudions l'écoulement stationnaire d'un fluide incompressible du modèle à huit paramètres d'Oldroyd dans un régime annulaire entre deux sphères, souvent appelé écoulement de Couette. La sphère intérieure tourne avec une vitesse Ω autour d'un axe z passant par le centre des deux sphères, pendant que la sphère extérieure est gardée immobile. Nous supposons que les effets viscoélastiques sont plus importants que les effets inertiaux, de telle sorte que ces derniers sont négligés dans l'équation pour le moment. Nous obtenons une solution analytique via l'expansion des variables dynamiques en séries de puissance du temps retardé. Le premier terme en vitesse décrit la rotation newtonienne autour de l'axe z. Le terme de premier ordre décrit un écoulement secondaire représenté par une fonction de flot qui divise la région d'écoulement en quatre parties symétriques. Le terme de deuxième ordre est la contribution viscoélastique à l'écoulement primaire visqueux. L'approximation de premier ordre dépend de la viscosité et de quatre paramètres du modèle. L'approximation de deuxième ordre dépend des huit paramètres viscométriques du modèle. Le torque sur la surface de la sphère extérieure a un terme additionnel dû à la viscoélasticité et dépend

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de tous les paramètres du fluide étudié. Pour un fluide d'Oldroyd à huit paramètres, ce terme augmente le torque primaire, mais dans le cas de fluides de plus grande élasticité, les contributions au torque peuvent être positives ou négatives selon la valeur des paramètres viscométriques.

[Traduit par la Rédaction]

1. Introduction

Fluid flow in the annular region between two rotating objects has attracted attention for the last few decades in many branches of industry and technology [1–5].

Much theoretical and experimental work had been done on the viscous flow between two eccentric spheres. Jeffery [6], Stimson and Jeffery [7] solved the stationary rotational viscous flow in the so called axisymmetrical case, where the rotation takes place about the common diameter of the two spheres. These authors employed the bispherical system of coordinates, which appears to be the most appropriate one. Majumdar [8] considered the non-axisymmetrical problem of separate spheres in an incompressible viscous fluid when one of the spheres rotates slowly about an axis perpendicular to their line of centers and the other sphere remains at rest. Due to the complications associated with the non-axisymmetry of the problem, the solution is obtained by applying special approximation methods. Munson [9] solved the axisymmetrical case for stationary slow viscous flow using spherical polar coordinates instead of the bispherical coordinates. Due to the complications that appear, where the used system of coordinates is not appropriate for the boundary conditions, a series of computational approximations are done. Menguturk and Munson [10] construct a device to realize experimentally the results obtained in the previous paper. They compare the theoretical values of the torques on the outer nonmoving sphere with those measured experimentally. Recently, Abu-El Hassan et al.[11] studied the flow of a viscoelastic second-order fluid between two eccentric spheres using bispherical coordinates. They calculated the first-order velocity field as well as the distribution of stresses and total forces and torques acting on the outer stationary sphere.

Numerous works have dealt with the viscous flow in the gap between two concentric spheres both theoretically and experimentally. Wimmer [12] study this problem experimentally and shows how the flow field takes place in the two cases; namely, when one of the two spheres is rotating while the other is at rest and in the other case in which both of the two spheres are rotating . However, for Newtonian spherical Couette flow there are two parameters that affect various flow modes; namely, the Reynolds number Re and the clearance ratio $B = (R_2 - R_1)/R_1$ where R_1 and R_2 are the radii of the inner and outer spheres, respectively. If the inner sphere is rotating and the outer sphere is held stationary, various types of flow modes such as spiral Taylor–Gortler (TG) vortices and multiple shear waves with different wave numbers and rotational frequencies have been observed experimentally, Nakabayashi [13] and Nakabayashi et al. [14, 15]. On the other hand, if only the outer sphere is rotating, the flow is hydrodynamically stable and no complex flow modes appear, Nakabayashi [16]. A transition to turbulent flow directly from the laminar stable flow is observed when the Re is increased without developing secondary flow; i.e., no TG vortex types modes appear.

In a series of important papers, Yamaguchi et al. [17–19] studied the spherical Couette flow of viscoelastic fluids. In Part I [17], an experimental set-up is constructed where the inner sphere is rotated and the outer sphere is kept stationary. The velocity field and the torque characteristics associated with the transition of the flow modes in different gap widths, specially the first flow-mode classification or the critical rotational Re, for polyacrylamide (PAA)–water solutions of low concentrations are investigated. In supercritical flows two distinct regions, elastic and inertial are formed in the spherical gap and spiral TG vortices have been observed in the inertial region. In Part II [18], numerical investigation is performed using the Giesekus model, and Oldroyd-B model as a special case, for low-to-moderate Re. The flow mode changes from weak secondary flow to the appearance of TG vortices. The study reveals that the shear-thinning effect on the shear viscosity strongly influences the flow characteristics in the equatorial region in the Giesekus model. In Part III [19], the authors use the same procedure as in ref. 17 but with the outer sphere rotating only. Two new flow modes, namely, the traveling cell (TC) mode and the TC+ twin vortices mode are observed in the experiment. Using the Giesekus model, the TC mode is investigated numerically. It is shown that TC is a toroidal vortex generated in the polar region and has a periodic nature. Moreover, the vortices that comprise the TC mode are very weak and they do not affect the torque strongly.

In the present paper, the flow field of an incompressible Oldroyd 8-constant fluid in the annular region between two concentric spheres is investigated. The inner sphere rotates with a uniform velocity Ω about the z-axis centered in the origin of the system and the outer sphere is at rest. For the sake of investigating the accuracy of the approximation used in solving the present problem, the results obtained are compared with the numerical and experimental data given in the literature for describing the shear thinning of the Oldroyd 8-constant model.

2. Formulation of the problem

A viscoelastic fluid moves in the annular space between two concentric spherical shells of radii R_1 and R_2 ($R_2 > R_1$). The motion is due to the rotation of the inner sphere with angular velocity Ω about the z-axis while the outer sphere is kept at rest.

2.1. Dimensionless variables

In the present work, for the dimensional variable quantities; namely, the length $\tilde{r}$, velocity $\boldsymbol{v}$, deformation tensor d, stress tensor t, pressure p, and the stream function ψ, it is more convenient to introduce the following dimensionless variables in terms of nondimensional spherical polar coordinates (r, θ, φ)

$$r = \frac{\tilde{r}}{R_1}, \quad \boldsymbol{V} = \frac{\boldsymbol{v}}{\Omega R_1} = (U\hat{r}, V\hat{\theta}, W\hat{\varphi}), \quad \mathsf{D} = \frac{\mathsf{d}}{\Omega}, \quad P = \frac{p}{\eta_0 \Omega}, \quad \Psi = \frac{\psi}{R_1^3 \Omega}, \quad \mathsf{T} = \frac{\mathsf{t}}{\eta_0 \Omega} \tag{2.1}$$

where, the nondimensionality is obtained by using R_1, Ω, and $\eta_0\Omega$ as the characteristic length, time, and stress, respectively.

2.2. Constitutive equation

To get a comprehensive idea about the behavior of a viscoelastic fluid, we adopted the Oldroyd 8-constant model in the present work. The suggested constitutive equation is linear in the stresses alone and contains all possible terms quadratic in the stress components and the velocity-gradient components consistent with giving a symmetric stress tensor. Moreover, this model already presents a considerable simplification with respect to the general models of simple fluids. Hence, it can describes more rheological behaviors than the convected Jeffreys model. The Oldroyd 8-constant model represents one of the most general constitutive equations for the last four decades, Zmievski et al. [20], and it includes a number of frequently used constitutive equations as special cases [21–23](see Table (A.1) of Appendix A). The Oldroyd constitutive equation relates the stresses and the kinematic variables through the nondimensional equation [24]

$$\mathsf{T} = 2\mathsf{D} - \lambda\Big[\xi_1 \overset{\nabla}{\mathsf{T}} + \xi_3 (\mathsf{T}.\mathsf{D} + \mathsf{D}.\mathsf{T}) - 2\xi_4 \mathsf{D}.\mathsf{D} + \xi_5 (\mathrm{tr}\mathsf{T})\, \mathsf{D} - 2\overset{\nabla}{\mathsf{D}} + (\xi_6 \mathsf{T} : \mathsf{D} - 2\xi_7 \mathsf{D} : \mathsf{D})\, \mathsf{I}\Big] \tag{2.2a}$$

$$\lambda = \lambda_2 \Omega, \qquad \xi_i = \frac{\lambda_i}{\lambda_2} \quad \text{for} \quad i = 1, 3, 4, 5, 6, 7$$

$$\mathsf{T}_{\mathrm{exc}} = -P\mathsf{I} + \mathsf{T} \tag{2.2b}$$

where T, $\mathsf{D} = \frac{1}{2}\left[(\nabla \boldsymbol{V}) + (\nabla \boldsymbol{V})^T\right]$, and I are the extra-stress, the rate-of-deformation, and the unit tensors, respectively. The material constants η_o, and λ_1 and λ_2 are the viscosity, and the relaxation and retardation times, respectively, while $\lambda_3 \ldots \lambda_7$ are further material time constants, T_{exc} is the Cauchy stress tensor. The upper convected derivative for any symmetric tensor G is defined by

$$\overset{\nabla}{\mathsf{G}} = \frac{\partial \mathsf{G}}{\partial t} + \boldsymbol{V}.\nabla\mathsf{G} - \mathsf{G} \cdot \nabla \boldsymbol{V} - (\mathsf{G} \cdot \nabla \boldsymbol{V})^T$$

2.3. The continuity and momentum equations

The dimensionless velocity field may be written as

$$\boldsymbol{V} = [(U(r,\ \theta)\hat{r},\ V(r,\ \theta)\hat{\theta},\ W(r,\ \theta)\hat{\varphi})] \tag{2.3}$$

The continuity equation is satisfied identically by introducing a stream function defined by

$$\boldsymbol{U}_\perp = U\hat{r} + V\hat{\theta} = -\nabla \wedge \left(\frac{\Psi}{r\sin\theta}\hat{\varphi}\right) \tag{2.4}$$

We assume that the forces due to viscoelasticity are dominated such that the inertial term $\boldsymbol{V} \cdot \nabla \boldsymbol{V}$ is negligible in the momentum equation. Thus, for steady state the momentum equation; $-\nabla P + \nabla \cdot \mathsf{T} = 0$, is given by

$$-\nabla P + 2\nabla \cdot \mathsf{D} - \lambda \left\{ \begin{array}{l} \nabla \cdot \left[\xi_1 \overset{\nabla}{\mathsf{T}} + \xi_3(\mathsf{T.d} + \mathsf{D.T}) - 2\xi_4 \mathsf{D.\ D} + \xi_5(\mathrm{tr}\,\mathsf{T})\mathsf{D} - 2\ \overset{\nabla}{\mathsf{D}}\right] \\ -\nabla\left[\xi_6 \mathsf{T} :\ \mathsf{D} - 2\xi_7 \mathsf{D} : \mathsf{D}\right] \end{array} \right\} = 0 \tag{2.5}$$

The perturbation term, (2.2a), is abbreviated in (2.5) by the vector $\boldsymbol{\Lambda}$, (r, θ) and the scalar $G(r, \theta)$ defined by the expressions

$$\boldsymbol{\Lambda}\,(r,\theta) = \nabla \cdot \left[\xi_1 \overset{\nabla}{\mathsf{T}} + \xi_3(\mathsf{T.D} + \mathsf{D.T}) - 2\xi_4(\mathsf{D.D}) + \xi_5(\mathrm{tr}\,\mathsf{T})\mathsf{D} - 2\,\overset{\nabla}{\mathsf{D}}\right] \tag{2.6a}$$

$$G(r,\theta) = \left[\xi_6 \mathsf{T} : \mathsf{D} - 2\xi_7 \mathsf{D} : \mathsf{D}\right] \tag{2.6b}$$

Hence, the momentum equation, (2.5), can be factorized into the pair of equations

$$(i) \quad \nabla^2(W\hat{\varphi}) - \lambda\,\Lambda_3\hat{\varphi} = 0, \qquad \Lambda_3 = \hat{\varphi} \cdot \boldsymbol{\Lambda} \tag{2.7}$$

which reduces to the scalar equation

$$\frac{1}{r^2}\left[\partial_r\left(r^2 W_{,r}\right) + \partial_\theta\left[\frac{1}{\sin\theta}\partial_\theta\ (W\sin\theta)\right]\right] - \lambda\Lambda_3 = 0 \tag{2.8}$$

$$(ii) \quad \nabla^2 \boldsymbol{U}_\perp - \lambda\,\boldsymbol{\Lambda}_\perp - \nabla(P - G) = 0, \qquad \boldsymbol{\Lambda}_\perp = \boldsymbol{\Lambda} - \hat{\varphi}\Lambda_3 \tag{2.9}$$

which reduces, through the application of the curl-operators, to the scalar equation

$$\left[\partial_r^2 + \frac{\sin\theta}{r^2}\partial_\theta\left[\frac{1}{\sin\theta}\partial_\theta\right]\right]^2 \Psi - \lambda\sin\theta\,[\partial_r\,(r\Lambda_2) - \partial_\theta\Lambda_1] = 0 \tag{2.10}$$

$$\boldsymbol{\Lambda}_\perp = \Lambda_1\hat{r} + \Lambda_2\hat{\theta} \tag{2.11}$$

The components of $\mathbf{\Lambda}$ are given as

$$\Lambda_1 = \hat{r}.\mathbf{\Lambda} = r^{-2}\partial_r \Big[r^2(\xi_1 \overset{\nabla}{T}_{rr} - 2\overset{\nabla}{D}_{rr} + 2\xi_3(T_{rr}D_{rr} + T_{r\theta}D_{\theta r} + T_{r\varphi}D_{\varphi r})$$
$$-4\xi_4(D_{rr}^2 + D_{r\theta}^2 + D_{r\varphi}^2) + \xi_5\Gamma D_{rr})] + \frac{1}{r\sin\theta}\partial_\theta[\sin\theta(\xi_1 \overset{\nabla}{T}_{r\theta} - 2\overset{\nabla}{D}_{r\theta}$$
$$+\xi_3(T_{rr}D_{r\theta} + T_{r\theta}D_{\theta\theta} + T_{r\varphi}D_{\varphi\theta} + D_{rr}T_{r\theta} + D_{r\theta}T_{\theta\theta} + D_{r\varphi}T_{\varphi\theta})$$
$$-4\xi_4(D_{rr}D_{r\theta} + D_{r\theta}D_{\theta\theta} + D_{r\varphi}D_{\varphi\theta}) + \xi_5\Gamma D_{r\theta})] + \frac{1}{r}[2(\overset{\nabla}{D}_{\theta\theta} + \overset{\nabla}{D}_{\varphi\varphi}) - \xi_1(\overset{\nabla}{T}_{\theta\theta} + \overset{\nabla}{T}_{\varphi\varphi})$$
$$-2\xi_3(T_{\theta r}D_{r\theta} + T_{\theta\theta}D_{\theta\theta} + T_{\theta\varphi}D_{\varphi\theta} + T_{\varphi r}D_{r\varphi} + T_{\varphi\theta}D_{\theta\varphi} + T_{\varphi\varphi}D_{\varphi\varphi})$$
$$+4\xi_4(D_{\theta r}^2 + D_{\theta\theta}^2 + D_{\theta\varphi}^2 + D_{r\varphi}^2 + D_{\theta\varphi}^2 + D_{\varphi\varphi}^2) - \xi_5\Gamma(D_{\theta\theta} + D_{\varphi\varphi})] \quad (2.12a)$$

$$\Lambda_2 = \hat{\theta}.\mathbf{\Lambda} = r^{-3}\partial_r[r^3(\xi_1 \overset{\nabla}{T}_{r\theta} - 2\overset{\nabla}{D}_{r\theta}$$
$$+\xi_3(T_{rr}D_{r\theta} + T_{r\theta}D_{\theta\theta} + T_{r\varphi}D_{\varphi\theta} + D_{rr}T_{r\theta} + D_{r\theta}T_{\theta\theta} + D_{r\varphi}T_{\varphi\theta})$$
$$-4\xi_4(D_{rr}D_{r\theta} + D_{r\theta}D_{\theta\theta} + D_{r\varphi}D_{\varphi\theta}) + \xi_5\Gamma D_{r\theta})]$$
$$+\frac{1}{r\sin\theta}\partial_\theta[\sin\theta(\xi_1 \overset{\nabla}{T}_{\theta\theta} - 2\overset{\nabla}{D}_{\theta\theta} + 2\xi_3(T_{\theta r}D_{r\theta} + T_{\theta\theta}D_{\theta\theta} + T_{\theta\varphi}D_{\varphi\theta})$$
$$-4\xi_4(D_{r\theta}^2 + D_{\theta\theta}^2 + D_{\theta\varphi}^2) + \xi_5\Gamma D_{\theta\theta})] + \frac{\cot\theta}{r}(2\overset{\nabla}{D}_{\varphi\varphi} - \xi_1 \overset{\nabla}{T}_{\varphi\varphi}$$
$$-2\xi_3(T_{\varphi r}D_{r\varphi} + T_{\theta\varphi}D_{\varphi\theta} + T_{\varphi\varphi}D_{\varphi\varphi}) + 4\xi_4(D_{r\varphi}^2 + D_{\varphi\theta}^2 + D_{\varphi\varphi}^2) - \xi_5\Gamma D_{\varphi\varphi}) \quad (2.12b)$$

and

$$\Lambda_3 = \hat{\varphi}.\mathbf{\Lambda} = r^{-3}\partial_r[r^3(\xi_1 \overset{\nabla}{T}_{r\varphi} - 2\overset{\nabla}{D}_{r\varphi}$$
$$+\xi_3(T_{rr}D_{r\varphi} + T_{r\theta}D_{\theta\varphi} + T_{r\varphi}D_{\varphi\varphi} + D_{rr}T_{r\varphi} + D_{r\theta}T_{\theta\varphi} + D_{r\varphi}T_{\varphi\varphi})$$
$$-2\xi_4(D_{rr}D_{r\varphi} + D_{r\theta}D_{\theta\varphi} + D_{r\varphi}D_{\varphi\varphi}) + \xi_5\Gamma D_{r\varphi})] + \frac{1}{r\sin^2\theta}\partial_\theta[\sin\theta(\xi_1 \overset{\nabla}{T}_{\theta\varphi} - 2\overset{\nabla}{D}_{\theta\varphi}$$
$$+\xi_3(T_{\varphi r}D_{r\theta} + T_{\varphi\theta}D_{\theta\theta} + T_{\varphi\varphi}D_{\theta\varphi} + D_{\varphi r}T_{r\theta} + D_{\varphi\theta}T_{\theta\theta} + D_{\varphi\varphi}T_{\varphi\theta})$$
$$-4\xi_4(D_{\varphi r}D_{r\theta} + D_{\varphi\theta}D_{\theta\theta} + D_{\varphi\varphi}D_{\theta\varphi}) + \xi_5\Gamma D_{\theta\varphi})] \quad (2.12c)$$

where $\Gamma = T_{rr} + T_{\theta\theta} + T_{\varphi\varphi} = \mathrm{tr}(\mathsf{T})$.

The components of the tensors T, $\overset{\nabla}{\mathsf{T}}$, D, and $\overset{\nabla}{\mathsf{D}}$ are given in Appendix A.

2.4. Boundary conditions

The boundary conditions imposed on the functions W and Ψ are

$$W = \sin\theta, 0 \quad \text{at} \quad r = 1, a, \qquad a = \frac{R_2}{R_1} \quad (2.13a)$$

$$\Psi = \Psi_{,r} = 0, 0 \quad \text{at} \quad r = 1, a \quad (2.13b)$$

Then, using (2.8) and (2.10), the functions W and Ψ are determined.

3. Method of solution

Expanding the functions W, Ψ, D, T, and P in a power series of the parameter λ as follows:

$$A = \sum_{n=0} \lambda^n A^{(n)} \quad (3.1)$$

where A may represents any of the above functions. Consequently, the expansion of (2.2a), (2.8), and (2.10) are

$$\sum_{n=0}\lambda^n\Big[\mathsf{T}^{(n)} - 2\mathsf{D}^{(n)} + \lambda\sum_{k\le n}\Big[\xi_1\overset{\nabla(n-k,k)}{\mathsf{T}} - 2\overset{\nabla(n-k,k)}{\mathsf{D}} + \xi_3\left(\mathsf{T}^{(n-k)}.\mathsf{D}^{(k)} + \mathsf{D}^{(k)}.\mathsf{T}^{(n-k)}\right) - \xi_4\mathsf{D}^{(n-k)}.\mathsf{D}^{(k)} + \xi_5\left(\mathrm{tr}\mathsf{T}^{(n-k)}\right)\mathsf{D}^{(k)} + \left(\xi_6\mathsf{T}^{(n-k)} : \mathsf{D}^{(k)} - 2\xi_7\mathsf{D}^{(n-k)} : \mathsf{D}^{(k)}\right)\mathsf{I}\Big]\Big] = 0 \quad (3.2)$$

$$\sum_{n=0}\lambda^n\left[\frac{1}{r^2}\left[\partial_r\left(r^2W^{(n)}_{,r}\right) + \partial_\theta\left[\frac{1}{\sin\theta}\partial_\theta\left(W^{(n)}\sin\theta\right)\right]\right] - \lambda\sum_{k\le n}\Lambda_3^{(n-k,k)}\right] = 0 \quad (3.3)$$

and

$$\sum_{n=0}\lambda^n\left[\left[\partial_r^2 + \frac{\sin\theta}{r^2}\partial_\theta\left(\frac{1}{\sin\theta}\partial_\theta\right)\right]^2\Psi^{(n)} - \lambda\sin\theta\sum_{k\le n}\left[\partial_r\left(r\Lambda_2^{(n-k,k)}\right) - \partial_\theta\Lambda_1^{(n-k,k)}\right]\right] = 0 \quad (3.4)$$

The boundary conditions (2.13a) and (2.13b) can be written as

$$W^{(k)} = \delta_{0k}\sin\theta, 0 \quad \text{for} \quad r = 1, a \quad (3.5a)$$

$$\Psi^{(k)} = \Psi^{(k)}_{,r} = 0, 0 \quad \text{for} \quad r = 1, a \quad (3.5b)$$

Note: The expansion parameter, i.e., $\lambda = \lambda_2\Omega$ *is much smaller than unity for the range of* $\Omega \approx 10^2$ *since* λ_2 *is of the order* "10^{-2}–10^{-4} s" *according to the values quoted in the literature* [20].

4. Solutions of the successive set of equations

4.1. Zero-order approximation

Taking $n = 0$, the coefficients of λ^0 in (3.3) and (3.4) are

$$\frac{1}{r^2}\left[\partial_r\left(r^2W^{(0)}_{,r}\right) + \partial_\theta\left[\frac{1}{\sin\theta}\partial_\theta\left(W^{(0)}\sin\theta\right)\right]\right] = 0 \quad (4.1)$$

and

$$\left[\partial_r^2 + \frac{\sin\theta}{r^2}\partial_\theta\left(\frac{1}{\sin\theta}\partial_\theta\right)\right]^2\Psi^{(0)} = 0 \quad (4.2)$$

The solution of (4.1) that satisfies boundary condition (3.5a) is

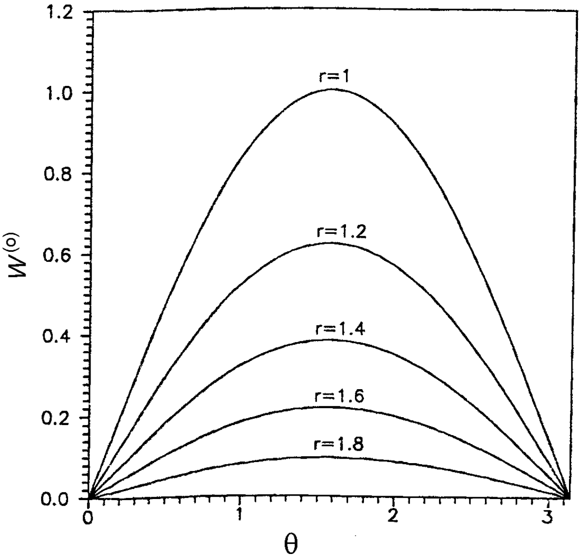
$$W^{(0)} = \left(a^3 - 1\right)^{-1}\left(a^3r^{-2} - r\right)\sin\theta \quad (4.3)$$

This velocity component $W^{(0)}$ is plotted in Fig. 1. The boundary conditions, (3.5b), imposed on (4.2), imply that

$$\Psi^{(0)} = 0 \quad (4.4)$$

Solutions (4.3) and (4.4) stand for a Newtonian fluid.

Fig. 1. Zero-order velocity field $W^{(0)}(r,\theta)$ as a function of θ where r is taken as a parameter.



4.2. First-order approximation

Letting $n = 1$, then the coefficients of λ in (3.3)–(3.4) deliver the following pair of equations

$$\frac{1}{r^2}\partial_r\left(r^2 W^{(1)}_{,r}\right) + \frac{1}{r^2}\partial_\theta\left[\frac{1}{\sin\theta}\partial_\theta\left(W^{(1)}\sin\theta\right)\right] - \lambda\Lambda_3^{(0,0)} = 0 \tag{4.5}$$

and

$$\left[\partial_r^2 + \frac{\sin\theta}{r^2}\partial_\theta\left[\frac{1}{\sin\theta}\partial_\theta\right]\right]^2 \Psi^{(1)} - \sin\theta\left[\partial_r\left(r\Lambda_2^{(0,0)}\right) - \partial_\theta\Lambda_1^{(0,0)}\right] = 0 \tag{4.6}$$

Using (2.12), the components $\Lambda_i^{(0,0)}$ for $i = 1, 2, 3$ are given by

$$\Lambda_1^{(0,0)} = \frac{1}{r^2}\partial_r\left[4r^2(\xi_3-\xi_4)(\overset{(0)}{D}_{r\varphi})^2\right] - \frac{1}{r}\left[\xi_1\overset{\nabla(0,0)}{T}_{\varphi\varphi} + 4(\xi_3-\xi_4)(\overset{(0)}{D}_{r\varphi})^2\right]$$

$$= 18\left[(\xi_1-1)-(\xi_3-\xi_4)\right]\left(\frac{a^3}{a^3-1}\right)^2 r^{-7}\sin^2\theta \tag{4.7a}$$

$$\Lambda_2^{(0,0)} = -\frac{\cot\theta}{r}\left[\xi_1\overset{\nabla(0,0)}{T}_{\varphi\varphi} - 4(\xi_3-\xi_4)(\overset{(0)}{D}_{r\varphi})^2\right]$$

$$= 36\left[(\xi_1-1)-(\xi_3-\xi_4)\right]\left(\frac{a^3}{a^3-1}\right)^2 r^{-7}\sin\theta\cos\theta \tag{4.7b}$$

and

$$\Lambda_3^{(0,0)} = 0 \tag{4.7c}$$

Using (3.5a), the solution of (4.5) is

$$W^{(1)}(r,\theta) = 0 \tag{4.8}$$

Substituting from (4.7a) and (4.7b) into (4.6) , we get

$$\left[\partial_r^2 + \frac{\sin\theta}{r^2}\partial_\theta\left[\frac{1}{\sin\theta}\partial_\theta\right]\right]^2 \Psi^{(1)} - 144\left[(\xi_1 - 1) - (\xi_3 - \xi_4)\right]\left(\frac{a^3}{a^3-1}\right)^2 r^{-7}\sin^2\theta\cos\theta = 0 \tag{4.9a}$$

Hence, the solution of the former equation with the boundary conditions

$$\Psi^{(1)} = \Psi^{(1)}_{,r} = 0, 0 \quad \text{for} \quad r = 1, a \tag{4.9b}$$

can be simplified into the form

$$\Psi^{(1)}(r,\theta) = -\left[(\xi_1 - 1) - (\xi_3 - \xi_4)\right]\left(\frac{a^3}{a^3-1}\right)^2 (a-r)^2(r-1)^2\Big[C_1 r + C_2 + C_3 r^{-1} + C_4 r^{-2} + r^{-3}\Big]\sin^2\theta\cos\theta \tag{4.10a}$$

where

$$\begin{aligned}
C_1 &= C_0^{-1}(3 + 12a + 15a^2 + 12a^3 + 3a^4)\\
C_2 &= C_0^{-1}(6 + 30a + 54a^2 + 54a^3 + 30a^4 + 6a^5)\\
C_3 &= C_0^{-1}(4 + 28a + 73a^2 + 100a^3 + 37a^4 + 28a^5 + 4a^6)\\
C_4 &= C_0^{-1}(2 + 16a + 52a^2 + 100a^3 + 100a^4 + 52a^5 + 16a^6 + 2a^7)\\
C_0 &= a^3(4 + 16a + 40a^2 + 55a^3 + 40a^4 + 16a^5 + 4a^6)
\end{aligned} \tag{4.10b}$$

The normalized stream-function $\Psi^{(1)}$ is shown in Fig. 2. Due to the presence of the parameters η_0 and λ_1 through λ_4, the flow field of $\Psi^{(1)}$ corresponds to that of an Oldroyd 5-constant fluid. As a special case, if $\lambda_3 = \lambda_4 = 0$, the field of $\Psi^{(1)}$ represents an Oldroyd-B fluid.

4.3. Second-order approximation

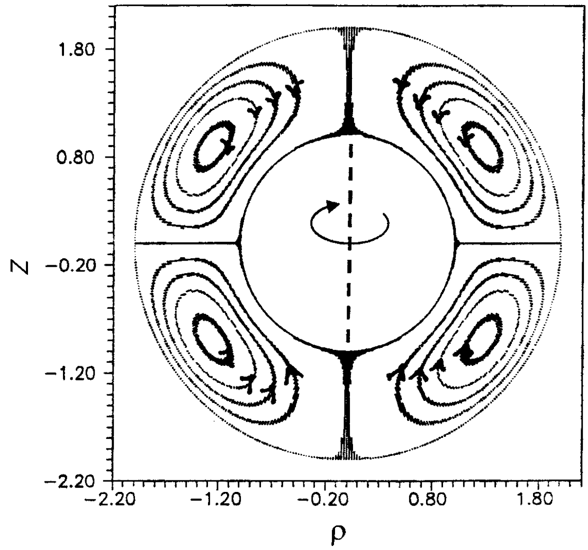
For $n = 2$, the coefficient of λ^2 in (3.3) and (3.4) are

$$\frac{1}{r^2}\partial_r\left(r^2 W^{(2)}_{,r}\right) + \frac{1}{r^2}\partial_\theta\left[\frac{1}{\sin\theta}\partial_\theta\left(W^{(2)}\sin\theta\right)\right] - \left(\Lambda_3^{(0,1)} + \Lambda_3^{(1,0)}\right) = 0 \tag{4.11}$$

and

$$\left[\partial_r^2 + \frac{\sin\theta}{r^2}\partial_\theta\left[\frac{1}{\sin\theta}\partial_\theta\right]\right]^2 \Psi^{(2)} - \sin\theta\left[\partial_r\left(r\Lambda_2^{(0,1)} + \Lambda_2^{(1,0)}\right) - \partial_\theta\left(\Lambda_1^{(0,1)} + \Lambda_1^{(1,0)}\right)\right] = 0 \tag{4.12}$$

Fig. 2. Streamlines of the normalized stream function $\Psi^{(1)}(r,\theta)$.



Using (2.12c), the components

$$\Lambda_3^{(0,1)} = r^{-3}\partial_r\left[r^3\left(\zeta_1\overset{\nabla}{T}{}^{(0,1)}_{r\varphi} - 2\overset{\nabla}{D}{}^{(0,1)}_{r\varphi} + \zeta_3\left(T^{(0)}_{r\varphi}D^{(1)}_{\varphi\varphi} + D^{(0)}_{r\varphi}T^{(1)}_{\varphi\varphi}\right) - 2\zeta_4\left(D^{(0)}_{r\varphi}D^{(1)}_{\varphi\varphi}\right)\right]\right. \\ + \frac{1}{r\sin^2\theta}\partial_\theta\left[\sin^2\theta\left[\zeta_1\overset{\nabla}{T}{}^{(0,1)}_{\varphi\varphi} - 2\overset{\nabla}{D}{}^{(0,1)}_{\varphi\varphi} + \zeta_3\left(T^{(0)}_{\varphi r}D^{(1)}_{r\theta} + D^{(0)}_{\varphi r}T^{(1)}_{r\theta}\right)\right.\right. \\ \left.\left. -4\zeta_4\left(D^{(0)}_{\varphi r}D^{(1)}_{r\theta}\right)\right]\right] \quad (4.13a)$$

$$\Lambda_3^{(1,0)} = r^{-3}\partial_r\left[r^3\zeta_1\overset{\nabla}{T}{}^{(1,0)}_{r\varphi} - 2\overset{\nabla}{D}{}^{(1,0)}_{r\varphi} + \zeta_3\left(T^{(1)}_{rr}D^{(0)}_{r\varphi} + D^{(1)}_{rr}T^{(0)}_{r\varphi}\right) - 2\zeta_4\left(D^{(1)}_{rr}D^{(0)}_{r\varphi}\right)\right. \\ \left. +\zeta_5\Gamma^{(1)}D^{(0)}_{r\varphi}\right] + \frac{1}{r\sin^2\theta}\partial_\theta\left[\sin^2\theta\left(\zeta_1\overset{\nabla}{T}{}^{(1,0)}_{\varphi\varphi} - 2\overset{\nabla}{D}{}^{(1,0)}_{\varphi\varphi} + \zeta_5\Gamma^{(1)}D^{(0)}_{\theta\varphi}\right)\right] \quad (4.13b)$$

then (4.11) becomes

$$\partial_r^2\left(rW^{(2)}\right) + \frac{1}{r}\partial_\theta\left[\frac{1}{\sin\theta}\partial_\theta\left(W^{(2)}\sin\theta\right)\right] = L\left(\frac{a^3}{a^3-1}\right)^3\left[\left(-\frac{156}{r^2}C_4 - \frac{90}{r^4}C_3\right.\right. \\ \left. - \frac{36}{r^7}C_1 - \frac{30}{r^9}C_2 + (36-162\beta')\frac{1}{r^{10}}\right)\sin^3\theta + \left(\frac{156}{r^2}C_4 + \frac{27}{r^4}C_3 + \frac{36}{r^7}C_1\right. \\ \left.\left. + \frac{72}{r^9}C_2 - \frac{108}{r^{10}}\right)\sin\theta\right] \quad (4.14a)$$

where

$$L = [(\xi_1 - 1) - (\xi_3 - \xi_4)]^2 \tag{4.14b}$$

$$\beta = \left[(\xi_1 - 1)(\xi_3 + \xi_5) + (\xi_3 - \xi_4)(\xi_1 - \xi_3 - \xi_5) - (\xi_7 - \xi_6)\left(\xi_1 - \xi_3 - \frac{3}{2}\xi_5 \right) \right] \tag{4.14c}$$

and

$$\beta' = \frac{\beta}{L} \tag{4.14d}$$

with the boundary conditions

$$W^{(2)} = 0, 0 \quad \text{at} \quad r = 1, a \tag{4.14e}$$

The solution of $W^{(2)}$ is given as

$$\begin{aligned} W^{(2)} = L\left(\frac{a^3}{a^3-1}\right)^3 \Bigg[\Bigg(& \left(\alpha_1 + \beta'\alpha_5\right) r^3 + 13C_4 + 9r^{-2}C_3 + \left(\alpha_2 + \beta'\alpha_6\right) r^{-4} - \frac{9}{2} r^{-5} C_1 \\ & - r^{-7}C_2 + \frac{9}{11}\left(1 - \frac{9}{2}\beta'\right) r^{-8} \Bigg) \sin^3\theta + \Bigg(\frac{4}{5}\left(-1 + \beta'\right)\alpha_1 r^3 + \left(\alpha_4 + \beta'\alpha_8\right) r \\ & - 26C_4 + \left(\alpha_3 + \beta'\alpha_7\right) r^{-2} + \frac{4}{5}\left(-1+\beta'\right)\alpha_2 r^{-4} + 4r^{-5}C_1 - \frac{8}{5}r^{-7}C_2 \\ & - \left(\frac{70}{3} + \frac{6}{11}\beta'\right) r^{-8} \Bigg) \sin\theta \Bigg] \end{aligned} \tag{4.15}$$

Using the boundary conditions, (4.14e), the constants $\alpha_1, \alpha_2, \ldots, \alpha_8$ can be determined (see Appendix B). The φ-component of the second-order velocity field $W^{(2)}$ contains all the parameters included in the Oldroyd 8-constant model that has been used.

To find the solution of (4.12) for the determination of $\Psi^{(2)}(r, \theta)$, we note from (2.12a) and (2.12b) the following:

$$\Lambda_1^{(0,1)} = \Lambda_1^{(1,0)} = \Lambda_2^{(0,1)} = \Lambda_2^{(1,0)} = 0 \tag{4.16a}$$

Therefore, (4.12) reduces to

$$\left[\partial_r^2 + \frac{\sin\theta}{r^2}\partial_\theta \left[\frac{1}{\sin\theta}\partial_\theta \right] \right]^2 \Psi^{(2)} = 0 \tag{4.16b}$$

with its boundary conditions

$$\Psi^{(2)} = \Psi^{(2)}_{,r} = 0, 0 \quad \text{for} \quad r = 1, a \tag{4.16c}$$

The solution of (4.16b) with the boundary conditions (4.16c) is given as

$$\Psi^{(2)}(r, \theta) = 0 \tag{4.17}$$

5. Determination of the torque on the outer sphere

The dimensionless torque M is given by

$$M = \int_0^{2\pi}\int_0^{\pi} -(T_{r\varphi})\big|_{R_1} R_2^3 \sin^2\theta \mathrm{d}\theta \mathrm{d}\varphi, \qquad r = a \tag{5.1}$$

with

$$T_{r\varphi} = T_{r\varphi}^{(0)} + \lambda T_{r\varphi}^{(1)} + \lambda^2 T_{r\varphi}^{(2)} + \lambda^3 T_{r\varphi}^{(3)} \tag{5.2a}$$

which reduces to

$$T_{r\varphi} = T_{r\varphi}^{(0)} + \lambda^2 T_{r\varphi}^{(2)} \tag{5.2b}$$

Finally, the dimensional torque $\tilde{M}$ is

$$\tilde{M} = \tilde{M}_0 + \tilde{M}_2 \tag{5.3a}$$

where the primary viscous torque $\tilde{M}_0$ is given by

$$\tilde{M}_0 = 8\pi\Omega\eta_0 a^3 \left(a^3 - 1\right)^{-1} \tag{5.3b}$$

and the dimensional viscoelastic contribution

$$\begin{aligned}\tilde{M}_2 = \left(2\pi\Omega^3\eta_0\right)\left(\frac{a^3}{a^3-1}\right)^3 \frac{1}{F}\Big\{L''\Big[&16(-8507 - 42535a - 127605a^3 - 315020a^4 \\ &-313608a^5 - 263571a^6 - 190268a^7 - 88969a^8 + 62070a^9 + 219806a^{10} \\ &+310651a^{11} + 313608a^{12} + 263571a^{13} + 198775a^{14} + 131504a^{15} + 65535a^{16} \\ &+ 21845a^{17} + 4369a^{18}\Big)\Big] + 144\beta''\Big[\Big(164 + 820a + 2460a^2 + 4759a^3 + 6575a^4 \\ &+7671a^5 + 8484a^6 + 9188a^7 + 10156a^8 + 11685a^9 + 13291a^{10} + 13919a^{11} \\ &+ 12867a^{12} + 9786a^{13} + 5705a^{14} + 2569a^{15} + 912a^{16} + 220a^{17} + 44a^{18}\Big)\Big]\Big\}\end{aligned} \tag{5.3c}$$

where

$$\begin{aligned}F = 55\left(-1 + a^3\right)^3 &\left(1 + a + a^2 + a^3 + a^4 + a^5 + a^6\right) \\ &\times\left(4 + 16a + 40a^2 + 55a^3 + 40a^4 + 16a^5 + 4a^6\right)\end{aligned}$$

and

$$\beta'' = \left[(\lambda_1 - \lambda_2)(\lambda_3 + \lambda_5) + (\lambda_3 - \lambda_4)(\lambda_1 - \lambda_3 - \lambda_5) - (\lambda_7 - \lambda_6)\left(\lambda_1 - \lambda_3 - \frac{3}{2}\lambda_5\right)\right]$$

and

$$L'' = [(\lambda_1 - \lambda_2) - (\lambda_3 - \lambda_4)]^2$$

The viscoelastic contribution depends on all the material parameters of the fluid under consideration.

The normalized curves for the torques $\tilde{M}_0$ and $\tilde{M}_2$ as well as for the stress components $T_{r\varphi}^{(0)}$ and $T_{r\varphi}^{(2)}$ versus "a" for different values of the material parameters of the fluid are shown in Figs. 3–7.

Fig. 3. The normalized viscous term $\tilde{M}_0$ versus a.

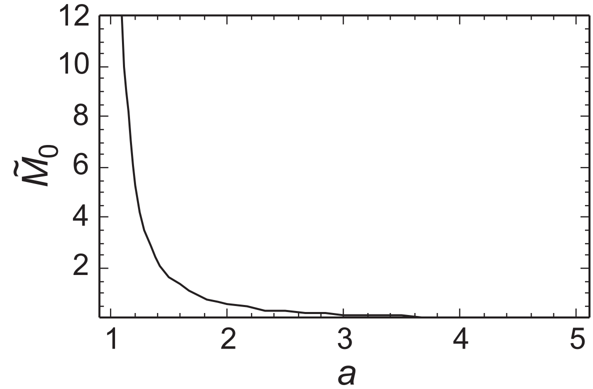
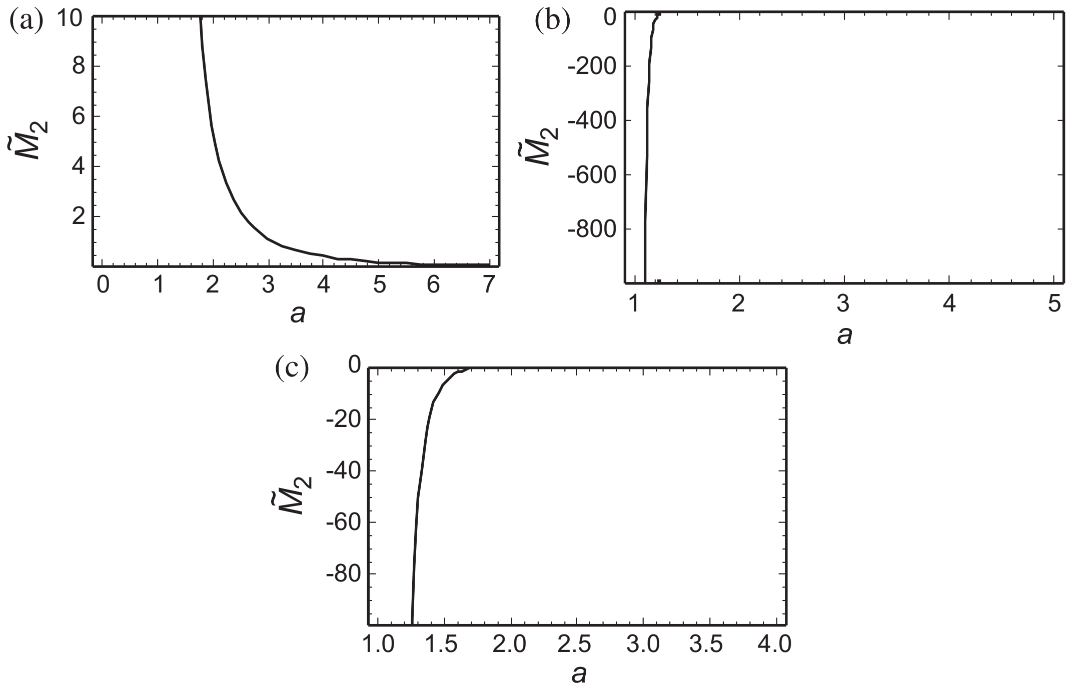


Fig. 4. (a) The normalized viscoelastic term $\tilde{M}_2$ versus a; $\beta = 0$ and $\lambda_1 - \lambda_2 = 0.5$. ($b$) The normalized viscoelastic term $\tilde{M}_2$ versus a; $\beta = -0.5$ and $\lambda_1 - \lambda_2 = 0.5$. ($c$) The normalized viscoelastic term $\tilde{M}_2$ versus a; $\beta = -1$ and $\lambda_1 - \lambda_2 = 0.5$.



6. Comparison with previous experimental and numerical works

To get an idea about the accuracy of the approximation used in solving the present problem, the results obtained are compared with the numerical and experimental results given in Nakamura et al. [25]. They calculated numerically the shear viscosity $\eta(\dot{\gamma})$ and the normal stress coefficient, $N_1(\dot{\gamma})$ using: (a) the Oldroyd 8-constant model, (b) the Giesekus model, and (c) the corotational Jeffreys model. Nakamura et al. compared their numerical results with the experimental data delivered by using three test fluids, namely,

Fig. 5. (*a*) The normalized viscoelastic term $\tilde{M}_2$ versus a; $\beta = 0$ and $\lambda_1 - \lambda_2 = 0.5$. (*b*) The normalized viscoelastic term $\tilde{M}_2$ versus a; $\beta = -0.5$ and $\lambda_1 - \lambda_2 = 0.5$. (*c*) The normalized viscoelastic term $\tilde{M}_2$ versus a; $\beta = -1$ and $\lambda_1 - \lambda_2 = 0.5$.

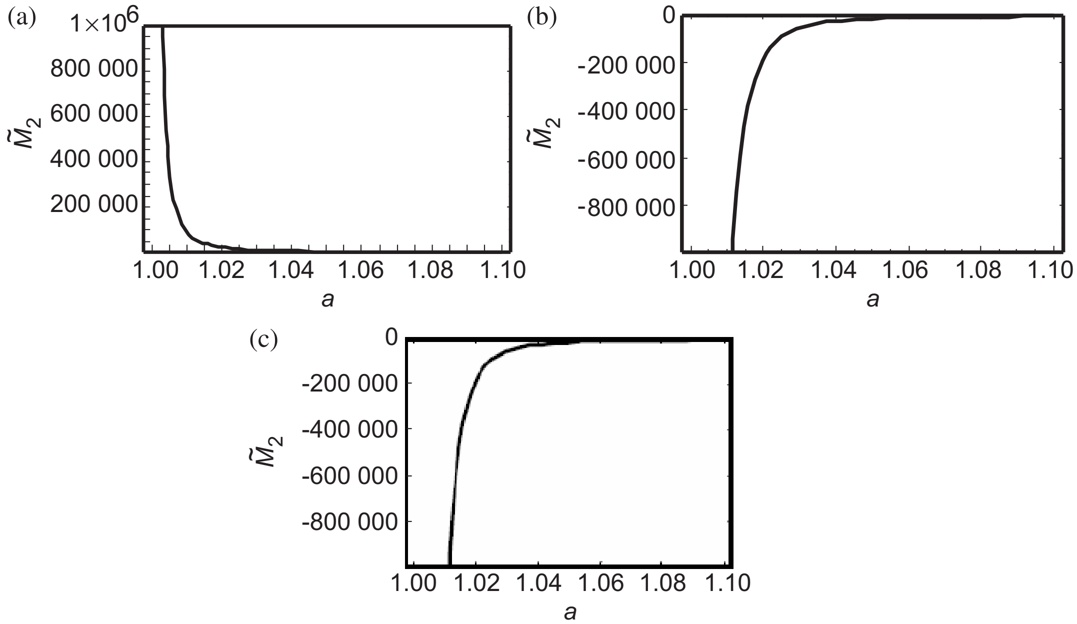
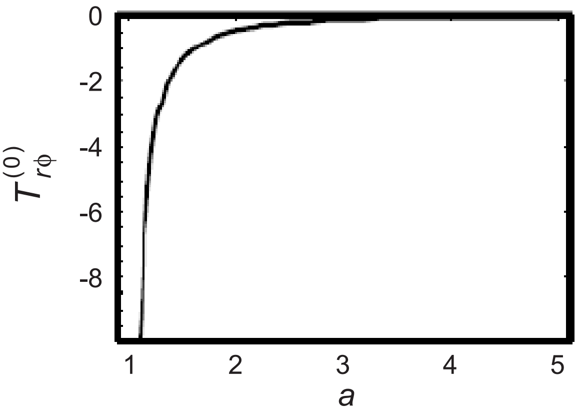


Fig. 6. The normalized stress term $T_{r\varphi}^{(0)}$ versus a; $\beta = -1$ and $\lambda_1 - \lambda_2 = 0.5$.



(*i*) 0.2 wt% aqueous solution of PAA (E-10 Allied Colloids (UK) Ltd),

(*ii*) 1.0 wt% aqueous solution of CMC (BDH Chemical Ltd), and

(*iii*) Mixture of CMC and PAA; 2 wt% aqueous solution of CMC and 0.5 wt% aqueous solution PAA, and the weight ratio of them in the mixture is CMC 0.84 wt%:PAA 0.2 wt%.

Concerning the Oldroyd model that is used in the present paper, Table 1 gives the numerical values of the 8-constants for each of these fluids.

Fig. 7. (*a*) The normalized stress term $T^{(2)}_{r\varphi}$ versus a; $\theta = \pi/2$, $\beta = 0$, and $\lambda_1 - \lambda_2 = 0.5$. (*b*) The normalized stress term $T^{(2)}_{r\varphi}$ versus a; $\theta = \pi/2$, $\beta = -0.5$, and $\lambda_1 - \lambda_2 = 0.5$. (*c*) The normalized stress term $T^{(2)}_{r\varphi}$ versus a; $\theta = \pi/2$, $\beta = -1$, and $\lambda_1 - \lambda_2 = 0.5$.

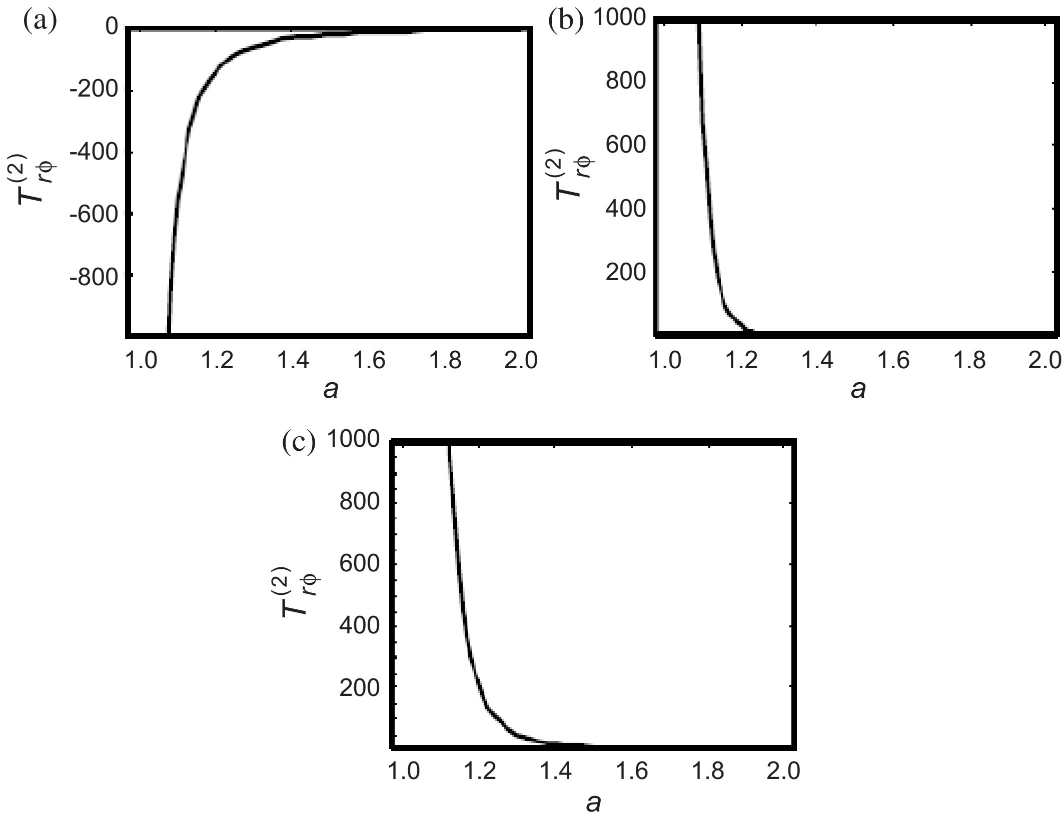


Table 1. Parameters for Oldroyd 8-constant model used in the comparison process [25].

Fluid	η_0(Pa s)	λ_1(s)	λ_2(s)	λ_3(s)	λ_4(s)	λ_5(s)	λ_6(s)	λ_7(s)
PAA 0.2 wt %	0.5	1.5	0.0001	0.1	0.1	1.05	0.05	0.05
CMC 1.0 wt %	2.5	2.0	0.0001	0.1	0.1	1.0	0.05	0.05
CMC.PAA mixture	2.5	2.0	0.0001	0.1	0.1	1.0	0.05	0.05

To perform the required comparison, the following formulas, [21, 25], are used for the determination of viscosity $\eta(\dot{\gamma})$ and the first-normal stress coefficient, $N_1(\dot{\gamma})$

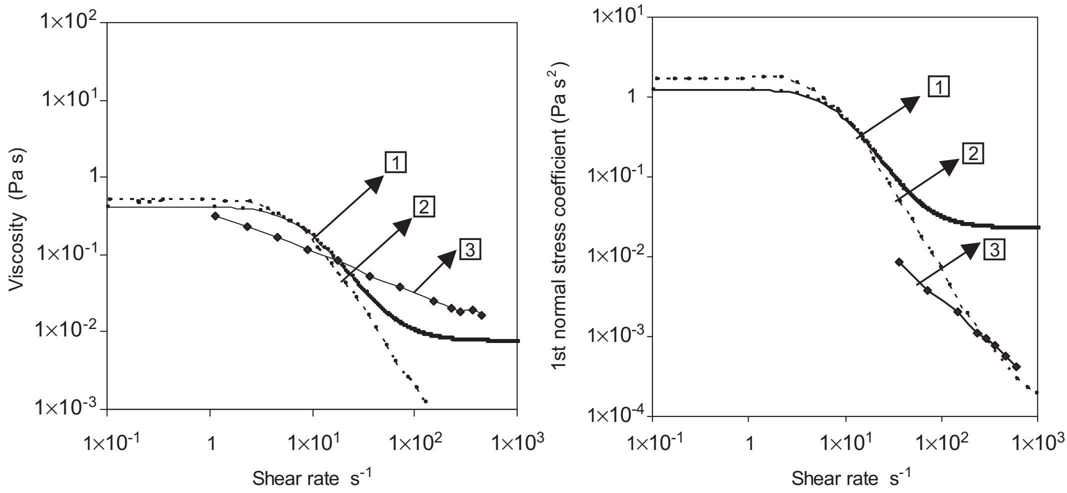
$$(i) \quad \eta(\dot{\gamma}) = \eta_0 \frac{1 + \sigma_2 \dot{\gamma}^2}{1 + \sigma_1 \dot{\gamma}^2} \tag{6.1}$$

$$(ii) \quad N_1(r, \theta) = 2[\lambda_1 \eta(\dot{\gamma}) - \lambda_2 \eta_0] \tag{6.2}$$

Here, $\dot{\gamma}$ is the shear rate tensor defined in terms of the rate of deformation tensor D, (2.2), by

$$\dot{\gamma}(r, \theta) = \sqrt{\frac{1}{2}(\mathsf{D} : \mathsf{D})} = \sqrt{2\left(D_{11}^2 + D_{22}^2 + D_{33}^2 + 2D_{12}^2 + 2D_{23}^2 + 2D_{13}^2\right)} \tag{6.3a}$$

Fig. 8. (*a*) $\eta(\dot{\gamma})$ versus $\dot{\gamma}$ for the Oldroyd 8-constant model as compared with experimental and numerical data for 0.2 wt% aqueous solution of PAA; (*b*) $N_1(\dot{\gamma})$ versus $\dot{\gamma}$ for the Oldroyd 8-constant model as compared with experimental and numerical data for 0.2 wt% aqueous solution of PAA.



where D_{ij} is the ij-component of D. σ_1 and σ_2 are, respectively, given by

$$\sigma_i = \lambda_i\,(\lambda_3 + \lambda_5) + \lambda_{i+2}\,(\lambda_1 - \lambda_3 - \lambda_5) + \lambda_{i+5}\,(\lambda_1 - \lambda_3 - (3/2)\lambda_5)\,, \qquad i = 1, 2 \tag{6.3b}$$

The shear rate $\dot{\gamma}$ as calculated from the solution

$$V = W^{(0)}\hat{\varphi} + \lambda U_{\perp}^{(1)} + \lambda^{(2)} W^{(2)}\hat{\varphi} + O(\lambda^3) \tag{6.4}$$

depends, of course, on the coordinates r and θ. Since the shear rate is varied in the gap space in the case of spherical Couette flow [19], so one can define $\dot{\gamma}$ by its average value, namely,

$$\tilde{\dot{\gamma}} = \frac{2\pi}{V} \int_{R_1}^{R_2} \int_{0}^{\pi} \dot{\gamma}(r,\theta) r^2 \sin\theta \mathrm{d}\theta \mathrm{d}r \tag{6.5}$$

where V represents the volume of the fluid contained between the two spherical shells.

The steady shear viscosity $\eta(\dot{\gamma})$ versus the shear rate $\dot{\gamma}$, for the three test fluids, are shown in Figs. 8*a*, 9*a*, and 10*a*, respectively. Similarly, the first-normal stress differences N_1 versus the shear rate $\dot{\gamma}$ are shown in Figs. 8*b*, 9*b*, and 10*b*, respectively.

7. Discussion

In the present work, the steady spherical Couette flow of an inertialless viscoelastic Oldroyd 8-constant fluid is investigated analytically. The flow field is created due to the rotation of the inner sphere R_1 with a uniform angular velocity Ω. The momentum equation is solved by using the method of successive approximation through the expansion of the dynamical variables in a power series of the retardation time λ. Herein, up to the second-order approximation the following results are found:

1. The resultant Newtonian velocity field $W^{(0)}(r,\theta)$ is a solid body rotation in the φ-direction [26], the velocity component is plotted in Fig. 1.

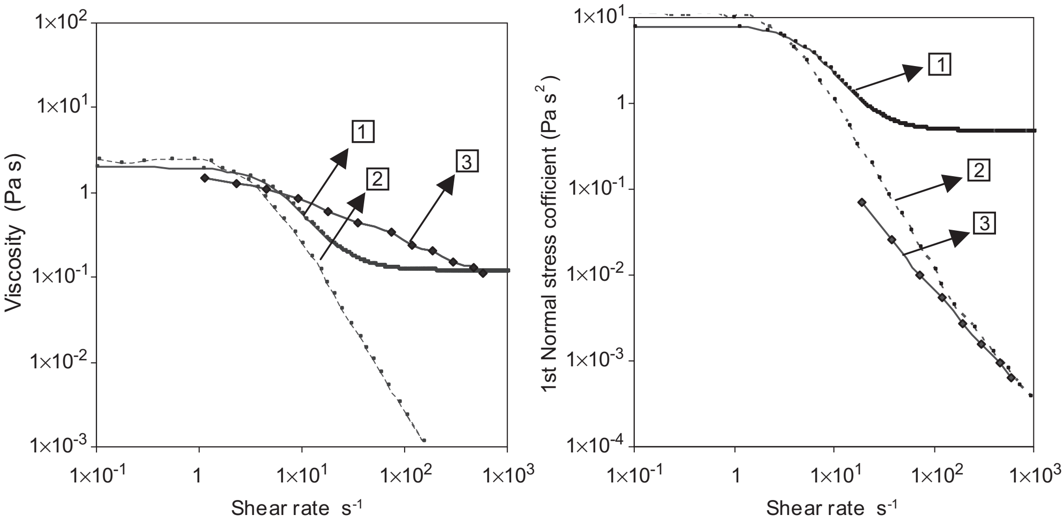


Fig. 9. (*a*) $\eta(\dot{\gamma})$ versus $\dot{\gamma}$ for the Oldroyd 8-constant model as compared with experimental and numerical data for 1.0 wt% aqueous solution of CMC; (*b*) $N_1(\dot{\gamma})$ versus $\dot{\gamma}$ for the Oldroyd 8-constant model as compared with experimental and numerical data for 1.0 wt% aqueous solution of CMC.

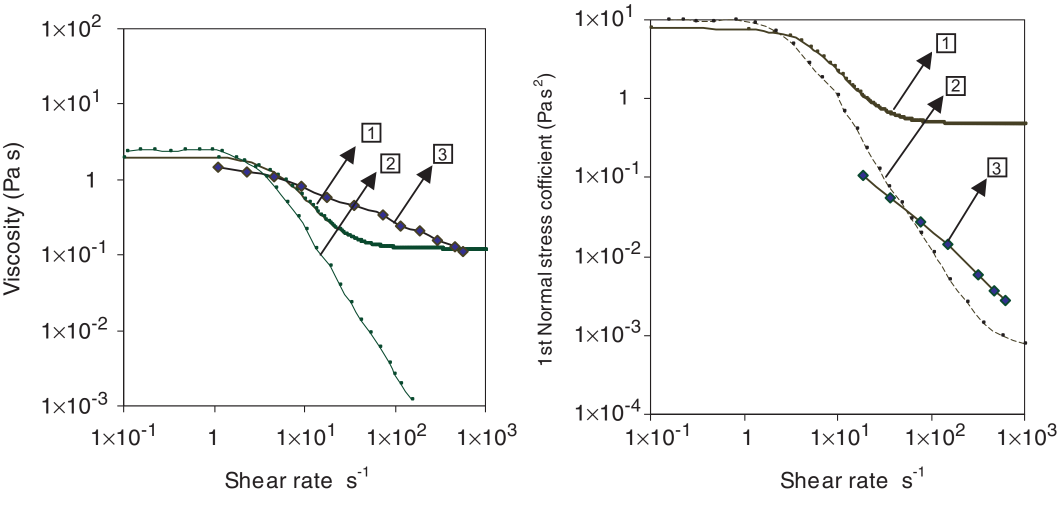


Fig. 10. (*a*) $\eta(\dot{\gamma})$ versus $\dot{\gamma}$ for the Oldroyd 8-constant model as compared with experimental and numerical data for CMC.PAA mixture; (*b*) $N_1(\dot{\gamma})$ versus $\dot{\gamma}$ for the Oldroyd 8-constant model as compared with experimental and numerical data for CMC.PAA mixture.

2. The first-order approximation results in a secondary flow represented by the stream-function $\Psi^{(1)}(r, \theta)$. Since this flow is affected by five parameters, namely, η_0 and λ_1 through λ_7 it is being in effect an Oldroyd 5-constant fluid [19]. As shown in Fig. 2, the streamlines of the normalized stream-function $\Psi^{(1)}(r, \theta)$ divides the annular region between the two spheres into four similar parts symmetric about the z-axis, which is the axis of rotation. On the basis of the last step, we note:

   (*i*) If $\lambda_3 + \lambda_4 = 0$, the stream function $\Psi^{(1)}(r, \theta)$ represents the secondary flow of an Oldroyd-

B model, which includes both an upper convected Maxwell fluid and a Newtonian fluid. Hence, as a special case of this flow field $\Psi^{(1)}$ if $\lambda_2 = 0$, then the Maxwell fluid flow is recovered as it is being a pure elastic fluid in constant shear flow.

(*ii*) If $\lambda_1 = 0$, then the flow field $\Psi^{(1)}$ is due to the flow of a second-order fluid, since $\lambda_4 = 0$, which means that the second normal stress difference $N_2 = 0$ in this case.

(*iii*) If the parameter $\eta_0 (\eta_0 = \eta_s + \eta_p)$ is the total viscosity with η_s and η_p are, respectively, the solvent and polymeric contributions to the total viscosity, λ_1 is the relaxation time of the fluid and $N_2 = 0$, then the Boger fluid is recovered [27].

(*iv*) Finally, the effective values of the parameters η_0, λ_1 and λ_2, $0 \leq \lambda_2 \leq \lambda_1$; $\lambda_2 \backslash \lambda_1 \leq 1$ with $\lambda_2 \backslash \lambda_1 = 1$ being the Newtonian limit, induce the recovered $\Psi^{(1)}$-flow.

3. The second-order approximation produces a φ-velocity component $W^{(2)}(r, \theta)$ in the direction of the primary flow. This component includes 8-parameters and hence it is viscoelastic contribution.

Therefore, in addition to the primary viscous torque $\tilde{M}_0$ [28], there exists a viscoelastic contribution $\tilde{M}_2$ that includes all the parameters of the Oldroyd 8-constant fluid that was used.

The normalized viscous torque $\tilde{M}_0$ versus the dimensionless ratio "a" is shown in Fig. 3. Inspection of (5.1) reveals that for fluids of lower elasticity; i.e., $\beta = 0$, which reduces to an Oldroyd-B fluid if $\lambda_3 = \lambda_4 = 0$, the normalized torque $\tilde{M}_2$ is enhanced relative to $\tilde{M}_0$ (Fig. 4*a*). A comparison of this result with the steady-state solution of Yamajuchi et al. (see ref. 18, p. 63) indicates this is the behavior of an Oldroyd-B fluid. For Oldroyd 8-constant fluids with $\beta = -0.5$ and $\beta = -1$, $\tilde{M}_2$ is diminished relative to $\tilde{M}_0$ (Figs. 4*b* and 4*c*). The same behaviors as the torque components are shown for fluids with $\beta = 0$, $\beta = -0.5$, and $\beta = -1$ in a narrow-gap width, which may be useful for lubrication theory (Figs. 5*a*–5*c*) respectively. Figures 6 and 7 show the behavior of the stress components $T_{r\phi}^{(0)}$ and $T_{r\phi}^{(2)}$ versus a for fluids with $\beta = -1$ (Fig. 6) and $\beta = 0$, $\beta = -0.5$, and $\beta = -1$ (Figs, 7*a*–7*c*) respectively. This insures the present results for $\tilde{M}_0$ and $\tilde{M}_2$.

Figures 8*a*, 9*a*, and 10*a* show that the present calculation gives better agreement with experimental work for $\eta(\dot{\gamma})$ than that obtained by Nakamura et al. [25]. On the other hand, Figs. 8*b*, 9*b*, and 10*b* show the opposite behavior for $N_1(\dot{\gamma})$.

8. Conclusion

The present work deals with the behavior of a viscoelastic fluid in a spherical Couette flow by using Oldroyd 8-constant model. The inner sphere is rotating with angular velocity Ω while the outer sphere is kept at rest.

At present, the solution of the momentum equation has been performed up to the second-order. The field variables are expanded in a power series in terms of the dimensionless retardation parameter λ. The zero-order velocity $W^{(0)}(r, \theta)$ is the Newtonian flow, which is pure rotation about the z-axis. The stream lines due to the stream function $\Psi^{(1)}(r, \theta)$ are the secondary flow, which is symmetric with respect to the axis of rotation; i.e., the z-axis. This flow divides the annular region between the two shells into four similar regions. The second-order approximation gives a viscoelastic contribution $W^{(2)}(r, \theta)$, in the φ-direction, which depends on the geometry of the annular region as well as on all the material parameters of the Oldroyd 8-constant fluid.

Moreover, the torque on the surface of the outer sphere is calculated. The calculations produced a viscoelastic contribution term $\tilde{M}_2$ proportional to the square of the retardation time superposed on the primary viscous term $\tilde{M}_0$, and depends on all the material parameters of the Oldroyd 8-constant fluid.

Finally, to describe the shear thinning of the Oldroyd 8-constant model in a steady simple shear flow in the present work; the shear viscosity $\eta(\dot{\gamma})$ and the first normal stress coefficient $N_1(\dot{\gamma})$ governing (6.1) and (6.2) are introduced on the basis of the velocity field obtained.

The results obtained are compared with the numerical and experimental results due to Nakamura et al. [25] for three test fluids. Since, the PAA and CMC solutions are both typical for viscosity, the mixture of PAA and CMC shows almost the same viscous property as a CMC 1.0 wt% solution, but is different from the CMC 1.0 wt% solution in the first normal stress in the present work.

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Appendix A. Constitutive models and spherical tensor components

(i) Constitutive equations included in Oldroyd 8-constant model as special cases are shown in Table A.1.

A1. Approximation method

The vector $\mathbf{\Lambda}$ defined by (2.12), or its components in (17), can be found through the relations:

The components of $\overset{\triangledown}{\mathsf{T}}$, $\overset{\triangledown}{\mathsf{D}}$ and $(\mathrm{tr}\mathsf{T})\mathsf{D}$

(ii) The tensors D and T have the components shown in Table A.2.

(iii) The components of an upper convected tensor A (which may be either D or T) are given by the

Table A.1. Constitutive equations included in Oldroyd 8-constant model as special cases.

No.	Name of the model	Constants included
1	Newtonian fluid (1-const.)	η_0 : $\lambda_i = 0$ for $i = 1, ..., 7$
2	Upper convected Maxwell fluid (2-const.)	η_0, λ_1 : $\lambda_i = 0$ for $i = 2, ..., 7$
3	Oldroyd-B fluid (3-const.)	$\eta_0, \lambda_1, \lambda_2$: $\lambda_i = 0$ for $i = 3, ..., 7$
4	Fluid of second order (3-const.)	$\eta_0, \lambda_2, \lambda_4$: $\lambda_i = 0$ for $i = 1, 3, 5, 6,7$
5	Gordon–Schowalter or Johnson–Segalman model (4-const.)	η_0 : $\lambda_2 = \frac{\eta_s}{\eta_0}\lambda_1,\ \lambda_3 = \xi\lambda_1,\ \lambda_4 = \xi\lambda_2,$ $\lambda_i = 0$ for $i = 5, 6, 7$ η_s is the solvent viscosity

Table A.2. Components of tensors D and T.

	$\hat{r}\hat{r}$	$\hat{\theta}\hat{\theta}$	$\hat{\varphi}\hat{\varphi}$	$\hat{r}\hat{\theta} + \hat{\theta}\hat{r}$	$\hat{r}\hat{\varphi} + \hat{\varphi}\hat{r}$	$\hat{\theta}\hat{\varphi} + \hat{\varphi}\hat{\theta}$
D_{ij}	D_{ij}	$\frac{U+V_{,\theta}}{r}$	$\frac{U+V\cot\theta}{r}$	$\frac{U_{,\theta}+rV_{,r}-V}{2r}$	$\frac{rW_{,r}-W}{2r}$	$\frac{W_{,\theta}-W\cot\theta}{2r}$
T_{ij}	T_{rr}	$T_{\theta\theta}$	$T_{\varphi\varphi}$	$T_{r\theta} = T_{\theta r}$	$T_{r\varphi} = T_{\varphi r}$	$T_{\theta\varphi} = T_{\varphi\theta}$

relations

$$\overset{\triangledown}{A}_{rr} = UA_{rr,r} + \frac{V}{r}A_{rr,\theta} - 2U_{,r}A_{rr} - 2\frac{U_{,\theta}}{r}A_{r\theta} \tag{A.1a}$$

$$\overset{\triangledown}{A}_{\theta\theta} = UA_{\theta\theta,r} + \frac{V}{r}A_{\theta\theta,\theta} - 2\frac{U+V_{,\theta}}{r}A_{\theta\theta} - 2\frac{V-rV_{,r}}{r}A_{r\theta} \tag{A.1b}$$

$$\overset{\triangledown}{A}_{\varphi\varphi} = UA_{\varphi\varphi,r} + \frac{V}{r}A_{\varphi\varphi,\theta} - 2\frac{U+V\cot\theta}{r}A_{\varphi\varphi} + 2\frac{W-rW_{,r}}{r}A_{r\varphi} + 2\frac{W\cot\theta - W_{,\theta}}{r}A_{r\varphi} \tag{A.1c}$$

$$\overset{\triangledown}{A}_{r\theta} = \overset{\triangledown}{A}_{\theta r} = UA_{r\theta,r} + \frac{V}{r}A_{r\theta,\theta} + \frac{U+V\cot\theta}{r}A_{r\theta} + \frac{V-rV_{,r}}{r}A_{rr} - \frac{U_{,\theta}}{r}A_{\theta\theta} \tag{A.1d}$$

$$\overset{\triangledown}{A}_{r\varphi} = \overset{\triangledown}{A}_{\varphi r} = UA_{r\varphi,r} + \frac{V}{r}A_{r\varphi,\theta} + \frac{U+V_{,\theta}}{r}A_{r\varphi} + \frac{W-rW_{,r}}{r}A_{rr} + \frac{W\cot\theta - W_{,\theta}}{r}A_{r\theta} - \frac{U_{,\theta}}{r}A_{\theta\varphi} \tag{A.1e}$$

$$\overset{\triangledown}{A}_{\theta\varphi} = \overset{\triangledown}{A}_{\varphi\theta} = UA_{\theta\varphi,r} + \frac{V}{r}A_{\theta\varphi,\theta} + U_{,r}A_{\theta\varphi} + \frac{W\cot\theta - W_{,\theta}}{r}A_{\theta\theta} + \frac{W-rW_{,r}}{r}A_{r\theta} + \frac{V-rV_{,r}}{r}A_{r\varphi} \tag{A.1f}$$

(iv) The components of the divergence of $\overset{\triangledown}{\mathsf{A}}$ are

$$(\nabla.\overset{\triangledown}{\mathsf{A}})_r = r^{-2}\partial_r(r^2\overset{\triangledown}{A}_{rr}) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta\overset{\triangledown}{A}_{r\theta}) - \frac{\overset{\triangledown}{A}_{\theta\theta} + \overset{\triangledown}{A}_{\varphi\varphi}}{r} \tag{A.2a}$$

$$(\nabla.\overset{\triangledown}{\mathsf{A}})_\theta = r^{-3}\partial_r(r^3\overset{\triangledown}{A}_{r\theta}) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta\overset{\triangledown}{A}_{\theta\theta}) - \frac{\overset{\triangledown}{A}_{\varphi\varphi}\cot\theta}{r} \tag{A.2b}$$

$$(\nabla.\overset{\triangledown}{\mathsf{A}})_\varphi = r^{-3}\partial_r(r^3\overset{\triangledown}{A}_{r\varphi}) + \frac{1}{r\sin^2\theta}\partial_\theta(\sin^2\theta\overset{\triangledown}{A}_{\theta\varphi}) \tag{A.2c}$$

Appendix B. Constants in (4.15)

The constants α_i, $i = 1, 2, \ldots, 8$ in (4.15) for the second-order viscoelastic contribution term $W^{(2)}(r, \theta)$ are

$$\alpha_1 = -\frac{1}{\left(a^{-4} - a^3\right)} \left[\left(-\frac{9}{11}\left(a^{-4} - a^3\right) + \frac{9}{11}a^{-8} - \frac{9}{11}a^3 \right) + \left(\frac{9}{2}\left(a^{-4} - a^3\right) - \frac{9}{2}a^{-5} + \frac{9}{2}a^3 \right) C_1 \right.$$
$$+ \left(\left(a^{-4} - a^3\right) - a^{-7} + a^3 \right) C_2 + \left(-\left(a^{-4} - a^3\right) + 9a^{-2} - 9a^3 \right) C_3$$
$$\left. + \left(-13\left(a^{-4} - a^3\right) + 13 - 13a^3 \right) C_4 \right]$$

$$\alpha_2 = -\frac{1}{\left(a^{-4} - a^3\right)} \left[\left(\frac{9}{11}a^{-8} - \frac{9}{11}a^3 \right) + \left(-\frac{9}{2}a^{-5} + \frac{9}{2}a^3 \right) C_1 + \left(-a^{-7} + a^3 \right) C_2 \right.$$
$$\left. + \left(9a^{-2} - 9a^3 \right) C_3 + \left(13 - 13a^3 \right) C_4 \right]$$

$$\alpha_3 = \frac{2}{15(-a^{-2} + a)} \left[\left(175\left(-a^{-2} + a\right) - 175a^{-8} + 175a^{-2} \right) \right.$$
$$+ \left(-30\left(-a^{-2} + a\right) + 30a^{-5} - 30a^{-2} \right) C_1 + \left(12\left(-a^{-2} + a\right) - 12a^{-7} + 12a^{-2} \right) C_2$$
$$+ \left(195\left(-a^{-2} + a\right) - 195a^{-2}\left(-1 + a^2\right) \right) C_4 + \left(6\left(-a^{-2} + a\right) - 6\left(-1 + a^5\right) \right) \alpha_1$$
$$\left. + \left(6\left(-a^{-2} + a\right) + 6a^{-4}\left(-1 + a^2\right) \right) \alpha_2 \right]$$

$$\alpha_4 = \frac{2}{15a^8\left(-a^{-2} + a\right)} \left[\left(175 - 175a^6 \right) + \left(-30a^3 + 30a^6 \right) C_1 + \left(12a - 12a^6 \right) C_2 \right.$$
$$\left. + \left(195a^6\left(-1 + a^2\right) \right) C_4 + \left(6a^6\left(-1 + a^5\right) \right) \alpha_1 - \left(6a^4\left(-1 + a^2\right) \right) \alpha_2 \right]$$

$$\alpha_5 = \frac{1}{\left(a^{-4} - a^3\right)} \left[\left(\frac{81}{22}\left(a^{-4} - a^3\right) \right) + \left(-\frac{81}{22}a^{-8} + \frac{81}{22}a^3 \right) \right]$$

$$\alpha_6 = \frac{1}{(a^{-4} - a^3)} \left[\left(\frac{81}{22}a^{-8} - \frac{81}{22}a^3 \right) \right]$$

$$\alpha_7 = \frac{1}{\left(-a^{-2} + a\right)} \left[\left(-\frac{6}{11}\left(-a^{-2} + a\right) + \frac{6}{11}a^{-8} - \frac{6}{11}a^{-2} \right) \right.$$
$$\left. + \left(\frac{4}{5}\left(-a^{-2} + a\right) - \frac{4}{5}a^{-4} + \frac{4}{5}a^{-2} \right) \alpha_6 + \left(\frac{4}{5}\left(-a^{-2} + a\right) + \frac{4}{5}a^{-2} - \frac{4}{5}a^3 \right) \alpha_5 \right]$$

$$\alpha_8 = \frac{-1}{\left(-a^{-2} + a\right)} \left[\left(\frac{6}{11}a^{-8} - \frac{6}{11}a^{-2} \right) + \left(-\frac{4}{5}a^{-4} + \frac{4}{5}a^{-2} \right) \alpha_6 + \left(\frac{4}{5}a^{-2} - \frac{4}{5}a^3 \right) \alpha_5 \right]$$